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Chinese Academy of Sciences Assignment
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凸分析与优化-作业

4 月 1 日作业

Question 1

设 $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. 证明以下结论成立:

1. $\ker(L) = (\operatorname{im}(L^*)^\perp$(课上已证), $(\ker(L))^\perp = \operatorname{im}(L^*)$;
2. $\ker(L^*) = (\operatorname{im}(L))^\perp, (\ker(L^*))^\perp = \operatorname{im}(L)$.

Solution. $\forall x \in \operatorname{im}(L^*)$, 存在 $y \in \mathbb{E}_2, x = L^*(y)$. 对于任意 $\overline{x} \in \ker(L)$,

$$\langle \overline{x}, L^*(y) \rangle = \langle L(x), y \rangle = 0.$$

故 $x \in (\ker(L))^\perp$. 这说明了

$$\operatorname{im}(L^*) \subset (\ker(L))^\perp.$$

由于

$$\mathbb{E}_1 = \operatorname{im} L^* \oplus \operatorname{im}(L^*)^\perp = \operatorname{im} L^* \oplus \ker(L) \subset (\ker(L))^\perp \oplus \ker(L) = \mathbb{E}_1,$$

得到

$$\mathrm{imL}^* \oplus \ker(L) = (\ker(L))^\perp \oplus \ker(L)$$

由于 $(\ker(L))^\perp \bigcap \ker(L) = \emptyset, \mathrm{imL}^* \bigcap \ker(L) = \emptyset$. 故必然有

$$(\ker(L))^\perp = \operatorname{im}(L^*)$$

由于 L 与 L^* 互为对偶, $\ker(L^*) = (\operatorname{im}(L))^\perp, (\ker(L^*))^\perp = \operatorname{im}(L)$ 由 $\ker(L) = (\operatorname{im}(L^*)^\perp$, $(\ker(L))^\perp = \operatorname{im}(L^*)$ 立得.

Question 2

设 $A \in \mathbb{S}^n_+$, 证明存在唯一的 $\boldsymbol{B} \in \mathbb{S}^n_+$ 使得 $B^2 = A$, 即 $B = A^{\frac{1}{2}}$. 这表明开根号运算在 $\mathbb{S}^n_+$ 中是良定的.

Solution. 存在性. 由于 $A \in \mathbb{S}_n^+$, 故存在 $T \in O(n)$, 使得 $T^{-1}AT = \Sigma_A$. Σ_A 是对角矩阵. 记

$$\Sigma_A = \operatorname{diag}\{\lambda_1, \cdots, \lambda_n\},\ \lambda_1 \geqslant 0, \cdots, \lambda_n \geqslant 0.$$

取

$$B = T \operatorname{diag}\{\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}\} T^{-1}.$$

即满足 $B^2 = A$. **唯一性**. 假如还存在 $C^2 = A$, 则 $(T^{-1}CT)^2 = \Sigma_A$, 易得 $T^{-1}CT \in \mathbb{S}_+^n$.

于是, 存在另一正交矩阵 $S \in O(n)$, 使得 $S^{-1}T^{-1}CTS = \operatorname{diag}\{\lambda_1', \cdots, \lambda_n'\}$, 而 $S^{-1}T^{-1}BTS = \operatorname{diag}\{\lambda_1, \cdots, \lambda_n\}$, 于是 B 和 C 可同时对角化. 故 $BC = CB$.

于是

$$B^2 - C^2 = (B + C)(B - C) = 0, \tag{1}$$

由于 $B + C \in \mathbb{S}_n^+$, 故 $|B + C| > 0$, 故 $B - C = 0$. 这说明了 B 的唯一性, 本题证毕.

Question 3

设 A, B 是 $\mathbb{E}$ 得非空闭子集, $\lambda, \mu \in \mathbb{R}$, A, B 中至少有一个有界. 证明 $\lambda A + \mu B$ 也是闭集, 并通过反例说明这里的有界性是必要的.

Solution. 不妨设 A 有界,$\forall\ z^* \in \overline{\lambda A + \mu B}$, 存在 $\{z_k = \lambda x_k + \mu y_k, x_k \in A, y_k \in B\}$, 使得 $\{z_k = \lambda x_k + \mu y_k\} \to z^*$. 由于 $\{x_k\}$ 有界, 故存在子列 $\{x_{k_m}\} \subset \{x_k\}$, 使得 $\{x_{k_m}\} \to x^* \in A$. 由此, $\{\mu y_{k_m}\} \to z^* - \lambda x^*$, 由 B 的闭性, $z^* - \lambda x^* \in \mu B$, 故 $z^* - \lambda x^* + \lambda x^* \in \lambda A + \mu B$, 故

$$\lambda A + \mu B = \overline{\lambda A + \mu B}.$$

即 $\lambda A + \mu B$ 是闭集.

反例: $A = \{n + \frac{1}{n}\}, B = \{-n\}$, 于是

$$A + B = \{k + \frac{1}{n}\}, k \in \mathbb{Z}, n \in \mathbb{N}^+.$$

易见 0 是 $A + B$ 的聚点, 但是 $0 \notin A + B$.

4 月 3 日作业

Question 1

证明: $\operatorname{epi}(\operatorname{cl}(f)) = \operatorname{cl}(\operatorname{epi}(f))$.

Solution. 记 $f:\mathbb{E}\to[-\infty,+\infty]$, 由于 $\mathrm{cl}(f)$ 是下半连续的, 因此 $\mathrm{epi}(\mathrm{cl}(f))$ 是闭的. 由于

$$\mathrm{cl}(f)\leqslant f,$$

故 $\mathrm{epi}(\mathrm{cl}(f))\supset\mathrm{epi}(f)$. 由 $\mathrm{epi}(\mathrm{cl}(f))$ 闭性, 知

$$\mathrm{epi}(\mathrm{cl}(f))\supset\mathrm{cl}(\mathrm{epi}(f)) \tag{2}$$

若 $\forall\,(x^*,\alpha^*)\in\mathrm{epi}(\mathrm{cl}(f))$, 有

$$\liminf_{x\to x^*} f(x^*)\leqslant\alpha^*.$$

即存在 α, 使得 $\exists\,\{x_k\}\to x^*, f(x_k)\to\alpha\leqslant\alpha^*$. 由于 $(x_k,f(x_k))\in\mathrm{epi}(f)$, 故 $(x^*,\alpha)\in\mathrm{cl}(\mathrm{epi}(f))$. 又 $\alpha^*\geqslant\alpha$, 故必然有 $(x^*,\alpha^*)\in\mathrm{cl}(\mathrm{epi}(f))$. 于是

$$\mathrm{epi}(\mathrm{cl}(f))\subset\mathrm{cl}(\mathrm{epi}(f)). \tag{3}$$

综合 (11) 和 (3), 得

$$\mathrm{epi}(\mathrm{cl}(f))=\mathrm{cl}(\mathrm{epi}(f)).$$

证毕.

4 月 8 日作业

Question 1

设 $A\in\mathbb{S}^n, b\in\mathbb{R}^n$, 二次函数 $q:\mathbb{R}^n\to\mathbb{R}$ 定义为

$$q(x)=\frac{1}{2}x^\top Ax+b^\top x$$

证明如下三条性质互相等价:

1. $\inf_{\mathbb{R}^n} q>-\infty$.
2. $A\succeq 0, b\in\mathrm{im}(A)$.
3. $\arg\min_{\mathbb{R}^n} q\neq\emptyset$.

Solution.

"(1) $\Rightarrow$ (2)": 由于 $\inf_{\mathbb{R}^n} q>-\infty$, 故一定存在 $x^*\in\mathbb{R}^n$, 使得 $x^*\in\arg\min_{\mathbb{R}^n} q$. 在 x^* 处任意施加扰动, 都有

$$q(x^*+\Delta x)-q(x^*)\geqslant 0. \tag{4}$$

由 (4) 可得,

$$\begin{aligned}
q(x^*+\Delta x)-q(x^*)=&(\Delta x)^\top(Ax^*+b)+(\Delta x)^\top A(\Delta x)\\
\xlongequal{\Delta x\to 0}&(\Delta x)^T(Ax^*+b)+o(\|\Delta x\|)\\
\geqslant&0.
\end{aligned}$$

对 $\forall \Delta x > 0$ 成立, 这说明 $Ax^*+b=0$, 即, $b \in \mathrm{im}(A)$ 另外, 当 $\|\Delta x\|$ 充分大时, $q(x^*+\Delta x)-q(x^*)$ 与 $\Delta x^\top A \Delta x$ 同号, 这说明, $\forall\ \Delta x$,

$$\Delta x^\top A \Delta x \geqslant 0. \tag{5}$$

这说明 $A \succeq 0$.

"(2) $\Rightarrow$ (3)": 由于 $b \in \mathrm{im}(A)$, 不妨记 $b = A\lambda$. 由于 $A \succeq 0$, 则

$$\begin{aligned} q(x) &= \frac{1}{2}x^\top Ax + b^\top x \\ &= \frac{1}{2}x^\top Ax + \lambda^\top Ax \\ &= \frac{1}{2}(x+\lambda)^\top A(x+\lambda) - \frac{1}{2}\lambda A \lambda^\top \\ &\geqslant \frac{1}{2}\lambda A \lambda^\top. \end{aligned}$$

且当 $x = -\lambda$ 时取等号. 于是 $\arg\min_{\mathbb{R}^n} q \supset \{-\lambda\} \neq \emptyset$.

"(3) $\Rightarrow$ (1)": 由 $q(x)$ 的适定性, 立得.

Question 2

证明: 设集合 $X \neq$, 给定一个包算子 $\mathrm{hull}: 2^X \to 2^X$, 可以确定一个包系统. 即证, $\mathcal{S} = \{\mathrm{hull}(S) \mid S \subset X\}$ 是一个包系统.

Solution. 根据教材 **命题 1.18** 知, 需验证 $\mathcal{S} = \{\mathrm{hull}(S) \mid S \subset X\}$ 满足包系统的两个条件.

1. $X \in \mathcal{S}$. 由

   $$X \subset \mathrm{hull}(X) \subset X$$

   立得.

2. 对任意非空 $\mathcal{A} \subseteq \mathcal{S}$,

   $$\bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(S) \subset \mathrm{hull}(\bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(S)) \subset \bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(\mathrm{hull}(S)) = \bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(S).$$

   故

   $$\bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(S) = \mathrm{hull}(\bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(S)).$$

   即

   $$\bigcap_{\mathrm{hull}(S)\in\mathcal{A}} \mathrm{hull}(S) \subset \mathcal{S} \tag{6}$$

   故 $\mathcal{S} = \{\mathrm{hull}(S) \mid S \subset X\}$ 是一个包系统.

4 月 10 日作业

Question 1

证明：设 $x_0, x_1, x_2, \cdots, x_k \in \mathbb{E}$, 则

$$x_0 + \operatorname{span}\{x_1 - x_0, \cdots, x_k - x_0\} = \operatorname{aff}\{x_0, \cdots, x_k\}. \tag{7}$$

特别地, 如果 $\{x_0, \cdots, x_k\}$ 包含 0, 则 $\operatorname{aff}\{x_0, \cdots, x_k\} = \operatorname{span}\{x_0, \cdots, x_k\}$.

Solution. 取 $\sum\limits_{i=0}^{k} a_i = 1$ 则

$$\begin{aligned} x_0 + \operatorname{span}\{x_1 - x_0, \cdots, x_k - x_0\} &= x_0 + \sum_{i=1}^{k} a_i(x_i - x_0) = (1 - \sum_{i=1}^{k} a_i)(x_0) + \sum_{i=1}^{k} a_i x_i \\ &= \sum_{i=0}^{k} a_i x_i = \operatorname{aff}\{x_0, \cdots, x_k\} \end{aligned} \tag{8}$$

故立得

$$x_0 + \operatorname{span}\{x_1 - x_0, \cdots, x_k - x_0\} = \operatorname{aff}\{x_0, \cdots, x_k\}. \tag{9}$$

若 $\{x_0, \cdots, x_k\}$ 包含 0，不妨记 $x_0 = 0$. 则由 (9), 代入 $x_0 = 0$, 有

$$\operatorname{span}\{x_0, \cdots, x_k\} = \operatorname{span}\{x_1 \cdots, x_k\} = x_0 + \operatorname{span}\{x_1 - x_0, \cdots, x_k - x_0\} = \operatorname{aff}\{x_0, \cdots, x_k\}. \tag{10}$$

本题证毕.

4 月 22 日作业

Question 1

Closure cl M of M , which in terms of hull operators is the intersection of all closed sets containing M, can also be written as

$$\begin{aligned} \text{cl } M &= \{x \mid \forall \varepsilon > 0, B_\varepsilon(x) \bigcap M \neq \emptyset\} \\ &= \{x \mid \exists \{x_k\} \subset M, \text{s.t. } \lim_{k \to \infty} x_k = x\} \end{aligned}$$

or

$$\text{cl } M = \bigcap_{\varepsilon > 0} M + \varepsilon B$$

The above definitions of the closed closure are equivalent, i.e., all of the above sets are the same.

Solution. "(1) $\Rightarrow$ (2) : $\bigcap\limits_{M \subset Q, Q\text{闭}} Q \subset \{x \mid \forall \varepsilon > 0, B_\varepsilon(x) \bigcap M \neq \emptyset\}$". 记 cl $M = \bigcap\limits_{M \subset Q, Q\text{闭}} Q$, 若存在 $x \in$ cl M, 但 $B_\varepsilon(x) \bigcap M \neq \emptyset$. 则 cl $M \setminus B_\varepsilon(x) \supset$ cl M, 矛盾!

"(2) $\Rightarrow$ (3) : $\{x \mid \forall \varepsilon > 0, B_\varepsilon(x) \bigcap M \neq \emptyset\} \subset \bigcap\limits_{\varepsilon > 0} M + \varepsilon B$". $\forall \varepsilon > 0, B_\varepsilon(x) \bigcap M \neq \emptyset$, 这说明, $x \in M + \varepsilon B$, 由 ε 任意性得证.

"$(3) \Rightarrow (1) : \bigcap\limits_{\varepsilon>0} M + \varepsilon B \subset \bigcap\limits_{M\subset Q, Q\text{闭}} Q$".

验证对于任意闭集 Q, 有 $\bigcap\limits_{\varepsilon>0} M + \varepsilon B \subset Q$. 用反证法, 假如存在 Q 是闭集, 但上述包含关系不成立, 也即存在 $\varepsilon > 0$, $B_\varepsilon(x) \in Q^c$. 即 $B_\varepsilon(x) \bigcap M = \emptyset$. 则 $x \notin M + \frac{\varepsilon}{2}B$. 矛盾!

Question 2

For a bounded set, the closed convex hull is also the convex hull of the closure of the said set. And give an example that in general $\overline{conv}S \neq \mathrm{conv}(\mathrm{cl}\, S)$.

Solution. 由 $\mathrm{cl}(\mathrm{conv})(S) \supseteq \mathrm{cl}(S)$, 且 $\mathrm{cl}(\mathrm{conv}(S))$ 是凸集, 易得

$$\mathrm{cl}(\mathrm{conv}(S)) \supseteq \mathrm{conv}(\mathrm{cl}(S)).$$

由 $\mathrm{cl}(S)$ 是紧集 (有界闭集), 故 $\mathrm{conv}(\mathrm{cl}(S))$ 是紧集, 自然是闭集. 故由 $\mathrm{conv}(\mathrm{cl}(S)) \supseteq \mathrm{conv}(S)$ 可得

$$\mathrm{cl}(\mathrm{conv}(S)) \subseteq \mathrm{conv}(\mathrm{cl}(S)).$$

故对于有界闭集 S, 有

$$\mathrm{cl}(\mathrm{conv}(S)) = \mathrm{conv}(\mathrm{cl}(S)).$$

例子: $S = (0,1) \cup x$ 轴. 则 $\mathrm{cl}(\mathrm{conv}(S)) = \mathbb{R} \times [0,1]$, $\mathrm{conv}(\mathrm{cl}(S)) = \{\mathbb{R} \times [0,1)\} \bigcup (0,1)$.

4 月 29 日作业

Question 1

设 $C_1, C_2 \subseteq \mathbb{E}$ 是凸集, 则 $\mathrm{ri}(C_1 + C_2) = \mathrm{ri}(C_1) + \mathrm{ri}(C_2)$.

Solution. 不妨设 $\dim(C_1) = \dim(C_2) = \dim(\mathbb{E})$

$$F(x_1, x_2) = x_1 + x_2$$

于是 $\mathrm{ri}(F(C_1 \times C_2)) = F(\mathrm{ri}(C_1 \times C_2))$

下证

$$\mathrm{ri}(C_1 \times C_2) = \mathrm{ri}(C_1) \times \mathrm{ri}(C_2).$$

任取 $(x_1, x_2) \in \mathrm{ri}(C)$. 存在 $\varepsilon > 0$, 使得 $B_\varepsilon((x_1, x_2)) \cap \mathrm{aff}(C_1 \times C_2) \subseteq C_1 \times C_2$. 而 $B_\varepsilon((x_1, x_2) \mid C_1) = B_\varepsilon(x_1)$,$B_\varepsilon((x_1, x_2) \mid C_2) = B_\varepsilon(x_2)$. 且易得 $\mathrm{aff}(C_1 \times C_2) = \mathrm{aff}(C_1) \times \mathrm{aff}(C_2)$. 于是 $B_\varepsilon(x_1) \times B_\varepsilon(x_2) \cap (\mathrm{aff}(C_1) \times \mathrm{aff}(C_2) = (B_\varepsilon(x_1) \cap \mathrm{aff}(C_1)) \times (B_\varepsilon(x_2) \cap \mathrm{aff}(C_2)) \subseteq C_1 \times C_2$. 这说明 $B_\varepsilon(x_1) \cap \mathrm{aff}(C_1) \subseteq C_1, B_\varepsilon(x_2) \cap \mathrm{aff}(C_2) \subseteq C_2$. 即 $(x_1, x_2) \in \mathrm{ri}(C_1) \times \mathrm{ri}(C_2)$. 反方向亦然.

于是，$\mathrm{ri}(F(C_1\times C_2)) = F(\mathrm{ri}(C_1\times C_2)) = F(\mathrm{ri}(C_1)\times \mathrm{ri}(C_2)) = \mathrm{ri}(C_1)+\mathrm{ri}(C_1)$, 得证

$$\mathrm{ri}(C_1+C_2) = \mathrm{ri}(C_1)+\mathrm{ri}(C_2).$$

Question 2

$\mathbb{R}^n_+, \mathbb{S}^n_+$ and $K = \{(x,t)\in\mathbb{R}^n\times\mathbb{R}_+ \mid |x|_2 \leqslant t\}$ are closed convex sets and all of them are self-dual.

Solution. $\forall x_1, x_2 \in \mathbb{R}^n_+$, $\lambda x_1 + (1-\lambda)x_2 \in \mathbb{R}^n_+, \lambda > 0$, 由于 R_+ 是闭集, 故 R^n_+ 是闭集.

$\forall A_1, A_2 \in \mathbb{S}^n_+$, $\lambda A_1 + (1-\lambda)A_2 \in \mathbb{S}^n_+, \lambda > 0$. 至于 S^n_+ 的闭性, 取一列 $A_1, \cdots, A_n, \cdots$, 记在某一矩阵范数 $\|\cdot\|$ 的意义下 $A_1,\cdots,A_n$ 收敛至 A. 则对于任意 $x\neq 0$, 有 $x^\top A_n x \to x^\top A x \geqslant 0$.
$\forall (x_1,t_1),(x_2,t_2)\in K$,

$$\lambda(x_1,t_1)+(1-\lambda)(x_2,t_2) = (\lambda x_1 + (1-\lambda)x_2) + (\lambda t_1 + (1-\lambda)t_2))$$

而 $|\lambda x_1 + (1-\lambda)x_2|_2 \leqslant |\lambda x_1|_2 + |(1-\lambda)x_2|_2 \leqslant \lambda t_1 + (1-\lambda)t_2$, 证毕.

闭性是显然的.

至于自对偶性,

$$(\mathbb{R}^n_+)^0 := \{d\in E \mid \langle d, x\rangle \leqslant 0, \forall x\in\mathbb{R}^n_+\}.$$

依此取 $e_1, e_2, \cdots, e_n$, 于是得到

$$(\mathbb{R}^n_+)^0 := \{(x_1,x_2,\cdots,x_n), x_1,\cdots,x_n \leqslant 0\} = -(\mathbb{R}^n_+)^0.$$

故 $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$.

设 $Y\in(\mathbb{S}^n_+)^*$, 若 $Y\notin(\mathbb{S}^n_+)^*$, 则存在 x 使得 $x^\top Y x < 0$. 记 $X := xx^\top \in \mathbb{S}^n_+$, 于是 $\langle Y, X\rangle < 0$, 矛盾! 于是 $(\mathbb{S}^n_+)^* \subseteq \mathbb{S}^n_+$.

反过来，任取 $X, Y\in\mathbb{S}^n_+$, 有谱分解 $X = \sum\limits_{i=1}^n \lambda_i q_i q_i^\top, \lambda_i \geqslant 0, i = 1,2,\cdots,n$. 因此

$$\langle Y, X\rangle = \mathrm{tr}(\sum_{i=1}^n \lambda_i q_i q_i^\top), \lambda_i \geqslant 0$$

这说明 $Y\in(\mathbb{S}^n_+)^*$, 于是 $\mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+)^*$. 综上

$\mathbb{S}^n_+ = (\mathbb{S}^n_+)^*$.

至于 $K = \{(x,t)\in\mathbb{R}^n\times\mathbb{R}_+ \mid |x|_2 \leqslant t\}$, 可写成等价形式 $K = \{(x,t) \mid (x,t)\text{与正半轴夹角不超过 45 度}\}$, 于是 $K^0 = \{(x,t) \mid (x,t)\text{与负半轴夹角不超过 45 度}\}$. 显然 $K^* = -K^0 = K$.

本题证毕.

Question 3

设 $S \subseteq \mathbb{E}$ 非空. 则

$$\operatorname{cone}(S) = C(S) = \mathbb{R}_+(\operatorname{conv}(S)) = (\operatorname{conv}(\mathbb{R}_+ S)$$

其中

$$C(S) := \{\sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N}, x_i \in S, \lambda_i \geqslant 0, i = 1, 2, \cdots, r\}$$

Solution.

$\operatorname{cone}(S) \subseteq C(S)$: 显然.

$C(S) \subseteq \mathbb{R}_+(\operatorname{conv}(S))$: 任取 $x \in C(S)$, 即

$$x = \sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N}, x_i \in S, \lambda_i \geqslant 0, i = 1, 2, \cdots, r$$

若 $x = 0$, 显然 $x \in \mathbb{R}_+(\operatorname{conv}(S))$. 若 $x \neq 0$, 则 λ_i 不全为 0, 即 $\lambda := \sum_{i=1}^{r} \lambda_i$. 则 $\frac{x}{\lambda} \in \operatorname{conv}(S)$.

$\mathbb{R}_+(\operatorname{conv}(S)) \subseteq (\operatorname{conv}(\mathbb{R}_+ S)$: $x \in \mathbb{R}_+(\operatorname{conv}(S)) = \lambda \sum_{i=1}^{r} \alpha_i x_i = \sum_{i=1}^{r} \alpha_i (\lambda x_i) \in \operatorname{conv}(\mathbb{R}_+ S)$

S $\supseteq \mathbb{R}_+ S$, 且 S 是凸集, 于是

$$\operatorname{cone}(S) \supseteq \operatorname{conv}(S)$$

本题证完.

Question 4

$K = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \mid \|x\|_2 \leqslant t\}$. For $x \in K$, compute $T_K(x)$ and $N_K(x)$.

当 $x \in \operatorname{int}(K)$, 于是 $B_\varepsilon(x) \in K$, 于是 $T_K(x) = \mathbb{R}^{n+1}$, $N_K(x) = 0$.

当 $x = 0$ 时, 有教材推论 2.4.9 有 $N_K(0) = K^o = -K$. 由于 $T_K(0) = (N_K(0))^o$, 于是 $N_K(0) = K$.

当 $x \in \operatorname{bd} K \setminus \{0\}$ 时, 有 $\|x\|_2 = t$. 由 $N_K(x) = \{v \in K^o \mid \langle v, x \rangle\rangle = 0\}$, 取 $(x', t') \in K$, 知

$$x'^{\top} x + t' t = 0$$

且 $\|x'\|_2 \leqslant -t'$. 故

$$-t' t = \|x' x\|_2 \leqslant \|x'\|_2 \|x\|_2 \leqslant t' t$$

于是得到 $x' = x$, $t' = -t$. 于是

$$N_K(x) = \mathbb{R}_+(x, -\|x\|)$$

则 $T_K(x) = (N_K(x))^0$.

5 月 6 日作业

Question 1

Let $C_1, C_2, \cdots, C_p \subseteq \mathbb{E}$, then

$$(\prod_{i=1}^{p} C_i)^\infty \subseteq \prod_{i=1}^{p} C_i^\infty.$$

Equality holds under either of the conditions.

(1) C_i is nonempty and convex for all i.

(2) There exists at most one i_0 such that C_{i_0} is unbounded.

Solution. $\forall v \in (\prod_{i=1}^{p} C_i)^\infty$, $\exists \{x_k\} \subseteq (\prod_{i=1}^{p} C_i)^\infty, \{t_k > 0\} \to 0, t_k x_k \to v\}$.

于是对于 $\forall i$, $\exists x_k(i) \in C_i, \{t_k > 0\} \to 0, t_k x_k(i) \to v(i)$. 故 $x_k(i) \in C_i^\infty, x_k \in \prod_{i=1}^{p} C_i^\infty$.

若 C_i 非空凸, 则由教材**命题 2.57**, $C^\infty = O^+(\mathrm{cl}(C))$. 故若 $v \in \prod_{i=1}^{p} C_i^\infty = \prod_{i=1}^{p} O^+(\mathrm{cl}(C_i))$. 则对于 $\forall x_i \in \mathrm{cl}(C_i), \lambda \geqslant 0, x_i + \lambda v_i \in \mathrm{cl}(C_i)$. 若说明 $\mathrm{cl}(\prod_{i=1}^{p} C_i) = \prod_{i=1}^{p} \mathrm{cl}(C_i)$, 则 $\forall (x_1, x_2, \cdots, x_p) \in cl(\prod_{i=1}^{p} C_i), \lambda \geqslant 0, x + \lambda(v_1, v_2, \cdots, v_p) \in \mathrm{cl}(\prod_{i=1}^{p} C_i)$. 故 $v \in O^+(\mathrm{cl}(\prod_{i=1}^{p} C_i)) = (\prod_{i=1}^{p} C_i)^\infty$. 这说明

$$(\prod_{i=1}^{p} C_i)^\infty \supseteq \prod_{i=1}^{p} C_i^\infty$$

结合 $(\prod_{i=1}^{p} C_i)^\infty \subseteq \prod_{i=1}^{p} C_i^\infty$, 可得 $(\prod_{i=1}^{p} C_i)^\infty = \prod_{i=1}^{p} C_i^\infty$.

下证 $\mathrm{cl}(\prod_{i=1}^{p} C_i) = \prod_{i=1}^{p} \mathrm{cl}(C_i)$.

方向 $\mathrm{cl}(\prod_{i=1}^{p} C_i) \subseteq \prod_{i=1}^{p} \mathrm{cl}(C_i)$ 是显然的.

对于 $\forall x \in \prod_{i=1}^{p} \mathrm{cl}(C_i)$, 则对于 $x_1 \in \mathrm{cl}(C_1)$, 存在 $x_1^{k^{(1)}} \in C_1$, 使得 $\{x_1^{k^{(1)}}\} \to x_1$ 同样的, 对于 $\forall (x_1^{k^{(1)}}, x_2) \in C_1 \times \mathrm{cl}(C_2)$, 可找到 $\{(x_1^{k^{(1)}}, x_2^{l^{(1)}})\} \to (x_1^{k^{(1)}}, x_2)$, 由对角线法则, 可找到一列 $\{(x_1^{k^{(1)}}, x_2^{k^{(2)}})\} \to (x_1, x_2)$. 于是 $(x_1, x_2) \in \mathrm{cl}(C_1) \times \mathrm{cl}(C_2)$ 依此进行下去, 可说明 $x \in \mathrm{cl}(\prod_{i=1}^{p} C_i)$. 故 $\prod_{i=1}^{p} \mathrm{cl}(C_i) \subseteq \mathrm{cl}(\prod_{i=1}^{p} C_i)$. 于是 $\mathrm{cl}(\prod_{i=1}^{p} C_i) = \prod_{i=1}^{p} \mathrm{cl}(C_i)$ 得证. 条件 (1) 证完.

下证条件 (2), 假如 $C_1, C_2, \cdots, C_p$ 均有界, 易得 $(\prod_{i=1}^{p} C_i)^\infty = \prod_{i=1}^{p} C_i^\infty = \{\mathbf{0}\}$. 若 C_j 无界, 但 $C_i (i \neq j)$ 有界. 由**命题 2.55**, 则 $(\prod_{i=1}^{p} C_i)^\infty = \{0\} \times \{0\} \times \cdots C_j^\infty \times \{0\} \times \cdots \{0\} = \prod_{i=1}^{p} C_i^\infty$.

Question 2

Let $T \in \mathcal{L}(E_1, E_2)$ and $C \in \mathbb{E}$ be closed such that $\ker\ T \cap C^\infty = 0$. Show $T(C^\infty) = (T(C))^\infty$.

Solution. $\forall y \in (T(C))^\infty$, $\exists \{T(x_k)\} \subset T(C), \{\lambda_k > 0\} \to 0$, s. t. $\lambda_k T(x_k) \to y$.

若 $\|\lambda_k x_k\| \to \infty$, 而 $\frac{\lambda_k x_k}{\|\lambda_k x_k\|} = \frac{x_k}{\|x_k\|} \to \overline{x}$, 得到 $T(\overline{x}) = 0$. 而 $\overline{x} = 1$, 与 $\ker\ T \cap C^\infty = 0$ 矛盾. 这样 $\{\lambda_k x_k\}$ 是有界子列, 故必然可以找到收敛子列 $\{\lambda_\alpha x_\alpha\} \subset \{\lambda_k x_k\}$. 由 C^∞ 是闭集, 则存在 $v \in C^\infty$. 使得 $\lambda_\alpha(x_\alpha) \to v$. 而 $y = \lim\limits_{k\to\infty} \lambda_k T(x_k) = T(\lambda_k x_k) = T(v) \in (T(C))^\infty$. 这说明 $(T(C))^\infty \subseteq T(C^\infty)$.

另一方面, 若 $y \in T(C^\infty)$, 则存在 $x_k \in C, \{\lambda_k > 0\} \to 0$, 使得 $T(\lambda_k \to x_k) \to y$. 而 $T(\lambda_k \to x_k) = \lambda_k T(x_k)$. 这说明 $y \in (T(C))^\infty$. 由此得 $T(C^\infty) \subseteq (T(C))^\infty$. 本题证毕.

5 月 8 日作业

Question 1

Let $K \subseteq \mathbb{E}$ be a cone. Then $K^{oo} = \overline{\text{conv}}K$.

由教材**注释 2.33**, K^{oo} 闭且凸. 显然有 $K^{oo} \supset \overline{\text{conv}}K$.

另一方面, 若 $\exists x \in K^{oo}$, 但 $x \notin \overline{\text{conv}}K$. 则有基本分离定理, 存在 $s \neq 0$, 满足

$$\langle s, x\rangle > \sup_{v \in \overline{\text{conv}}K} \langle s, v\rangle$$

故 $\langle s, x\rangle > 0$. 同时

$$\langle s, x\rangle > \sup_{v \in \overline{\text{conv}}K} \langle s, \lambda v\rangle$$

令 $\lambda \to \infty$, 得 $\langle s, v\rangle \leqslant 0,\ \forall v \in \overline{\text{conv}}K$.

于是 $s \in (\overline{\text{conv}}(K))^o \subset K^o$, 而 $x \in K^{oo} = (K^o)^o$, 于是必然有 $\langle r, x\rangle \leqslant 0$, 矛盾!

本题证毕!

5 月 13 日作业

Question 1

Let $s_1, s_2, \cdots, s_m$ be given in $\mathbb{R}^n$. Then the convex cone $K := \text{cone}\{s_1, \cdots, s_m\} = \{\sum\limits_{j=1}^m \alpha_j s_j :$ $\alpha_j \geqslant 0$ for$j = 1, 2, \cdots, m\}$ is closed.

Solution. 取 $\forall\ y \notin K$. 只有如下两种情况: $y \notin \text{span}\{s_1, s_2, \cdots, s_m\}$. 由于子空间是闭集, 故一定存在 $B_\varepsilon(y) \cap \text{span}\{s_1, s_2, \cdots, s_m\} = \emptyset$.

若 $y \in \text{span}\{s_1, s_2, \cdots, s_m\}$, 但 $y \notin \text{cone}\{s_1, s_2, \cdots, s_m\}$. 此时一定有 $\beta_i < 0$, 而 $y = \sum\limits_{j=1}^m \beta_j s_j$.

这时, 显然有 $B_{\frac{|\beta_i|}{2}}(\boldsymbol{\beta}) \cap \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \emptyset,\ \alpha_1, \alpha_2, \cdots, \alpha_m \geqslant 0, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$.

而 $[s_1, s_2, \cdots, s_n] B_{\frac{|\beta_i|}{2}}(\boldsymbol{\beta})$ 是包含 y 的开集, 且 $[s_1, s_2, \cdots, s_n] B_{\frac{|\beta_i|}{2}}(\boldsymbol{\beta}) \cap K = \emptyset$.

至此, 我们事实上证明了 K^c 是开集, 于是 K 是闭集.

Question 2

Let C be convex and compact, and $D \subset C$ such that $\overline{conv}(D) = C$,then $\operatorname{ext} C \subset \operatorname{cl} D$.

Solution. 若存在 $x \in \operatorname{ext}(C)$, 但 $x \notin \operatorname{cl}(D)$. 由于 C 是紧的, 所以 D 有界. 也即此时有 $\overline{\operatorname{conv}}(D) = \operatorname{conv}(\operatorname{cl}(D))$.

由 $x \in \operatorname{ext}\ C \subset \overline{\operatorname{conv}} D$. 则存在 $x_i \in \operatorname{cl}(D)$ 使得 $x = \sum\limits_i \lambda_i x_i, \sum\limits_i \lambda_i = 1$. 而 $x_i \in \operatorname{cl}(D) \subset C, x \in \operatorname{ext}(C)$, 这说明 $x_i = x \in \operatorname{cl}(D), \forall i$, 矛盾!

5 月 15 日作业

Question 1

Let $f : \mathbb{R} \to \mathbb{R} \bigcup \{+\infty\}$ and $I \subset \operatorname{dom} f$ be an open interval. Show the following:

(a) f is convex on I iff the slope function $x \to \frac{f(x)-f(x_0)}{x-x_0}$ is non-decreasing on $I \setminus \{x_0\}$.

(b) Let f is differentiable on I. Then f is convex on I. If f is non-decreasing on I, i.e. $f'(s) \leqslant f'(t)$ $(s \leqslant t)$

(c) Let f be twice differentiable on I. Then f is convex on I iff $f'' \geqslant 0.(\forall x \in I)$.

Solution. (a) '$\Rightarrow$': 只讨论 $x_2 \geqslant x_1 \geqslant x_0$. 由凸函数性质, 有

$$(x_2 - x_1) f(x) \leqslant (x - x_1) f(x_2) + (x_2 - x) f(x_1)$$

整理即得

$$\frac{f(x_2) - f(x_0)}{x_2 - x_0} \geqslant \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

其余情况只要按中间值展开均可证.

'$\Leftarrow$': 要证明, 对于任意 $x_1, x_2 \in I$, $\lambda \geqslant 0$. 则

$$f(\lambda x_1 + (1-\lambda) x_2) \geqslant \lambda f(x_1) + (1-\lambda) f(x_2). \tag{11}$$

不妨设 $x_2 \geqslant x_1$, 记 $\lambda x_1+(1-\lambda)x_2 := x_0$, 则由

$$\frac{f(x_2)-f(x_0)}{x_2-(\lambda x_1+(1-\lambda)x_2)} \geqslant \frac{f(x_1)-f(x_0)}{x_1-(\lambda x_1+(1-\lambda)x_2)}$$

可立得 (11).

(b): 由 (a) 立得:

$$f'(s) \leqslant \frac{f(t)-f(s)}{t-s} = \frac{f(s)-f(t)}{s-t} \leqslant \lim_{x\to t}\frac{f(x)-f(t)}{x-t} = f'(t)$$

(c): '⇒': 由 (b) 立得.

'⇐': 由中值定理, (c) 可推出 $\frac{f(x)-f(x_0)}{x-x_0}$ 非降. 进而由 (a) 的 '⇐' 立得 f 凸性.

Question 2

Let f be convex and lsc, $g:\mathbb{R}\to\mathbb{R}\bigcup\{+\infty\}$ be convex (and lsc) and increasing. Under the convention $g(+\infty)=\infty$ and $\lim\limits_{x\to+\infty} g(x)=+\infty$, $g\circ f$ is convex (and lsc).

Solution.. 保下半连续是显然的, 下证保凸.

$$\begin{aligned} g\circ f(\lambda x_1+(1-\lambda)x_2) \leqslant & g\circ(\lambda f(x_1)+(1-\lambda)f(x_2))(f\text{凸}+g\text{单增}) \\ \leqslant & \lambda g\circ(f(x_1))+(1-\lambda)g\circ(f(x_2))(g\text{凸}) \end{aligned}$$

证毕.

5 月 20 日作业

Question 1

证明 $f(X+H)-f(X)++\langle X^{-1},H\rangle-\frac{1}{2}\operatorname{tr}(X^{-1}HX^{-1}H)=o(\|H\|^2)$

Solution.

$$\begin{aligned} & f(X+H)-f(X)+\langle X^{-1},H\rangle-\frac{1}{2}\operatorname{tr}(X^{-1}HX^{-1}H) \\ \leqslant & -\log(det(I+X^{-\frac{1}{2}}HX^{-\frac{1}{2}}))+\operatorname{tr}(X^{-\frac{1}{2}}HX^{-\frac{1}{2}})-\operatorname{tr}(X^{-1}HX^{-1}H) \\ = & o(\operatorname{tr}(X^{-1}HX^{-1}H)) \\ = & o(\|H\|^2) \end{aligned}$$

Question 2

f 是超强制的, $g \in \Gamma$, 证明 $f+g$ 也是超强制的.

Solution. 由于 g 适定, 取 $x_0 \in \mathrm{ri}(\mathrm{dom}(f))$, 则由仿射下界定理, 存在 $g \in \mathbb{E}$

$$f(x) \geqslant f(x_0) + \langle g, x - x_0 \rangle$$

于是

$$\begin{aligned}\lim_{\|x\| \to +\infty} \frac{f(x)+g(x)}{\|x\|} &\geqslant \lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} + \frac{\langle g, x - x_0 \rangle}{\|x\|} \\ &\geqslant \lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} - \|g\| \\ &\geqslant +\infty.\end{aligned}$$

5 月 22 日作业

Question 1

证明 $\mathrm{epi}_<(f\#g) = \mathrm{epi}_<(f) + \mathrm{epi}_<(g)$

Solution. 先证 $\mathrm{epi}_<(f\#g) \subset \mathrm{epi}_<(f) + \mathrm{epi}_<(g)$.

若 $(x,a) \in \mathrm{epi}_<(f\#g)$, 则

$$\inf_{u \in \mathbb{E}} \{f(u) + g(x-u)\} < a$$

则存在 $u_1, \varepsilon > 0$, 使得 $f(u_1) + g(x - u_1) = \alpha - \varepsilon$. 于是 $(u_1, f(u_1) + \frac{\varepsilon}{2}) \in \mathrm{epi}_<(f)$, 且 $(u_2, g(u_2) + \frac{\varepsilon}{2}) \in \mathrm{epi}_<(g)$ 而 $(x, \alpha) = (u_1, f(u_1) + \frac{\varepsilon}{2}) + (u_2, g(u_2) + \frac{\varepsilon}{2})$, 故 $(x, \alpha) \in \mathrm{epi}_<(f) + \mathrm{epi}_<(g)$.

再证 $\mathrm{epi}_<(f\#g) \supset \mathrm{epi}_<(f) + \mathrm{epi}_<(g)$

$(x_1, \alpha_1) \in \mathrm{epi}_<(f)$,$(x_2, \alpha_2) \in \mathrm{epi}_<(g)$, 则

$$(f\#g)(x_1 + x_2) \leqslant f(x_1) + g(x_2) \leqslant \alpha_1 + \alpha_2.$$

这说明 $(x_1 + x_2, \alpha_1 + \alpha_2) \in \mathrm{epi}_<(f\#g)$. 即 $\mathrm{epi}_<(f\#g) \supset \mathrm{epi}_<(f) + \mathrm{epi}_<(g)$