Academy of Mathematics and Systems Science Mar. 28, 2024
Chinese Academy of Sciences Assignment
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Stochastic Operations Research-Homework 1

Question 1

Both E and F are two events. If $P(E) = 0.9$ and $P(F) = 0.8$, show that $P(EF) \geqslant 0.7$. In general, show that

$$P(EF) \geqslant P(E) + P(F) - 1. \tag{1}$$

This is known as Bonferroni's inequality.

Solution. In fact, $P(EF) = P(E) + P(F) - P(E \cup F)$. As $P(E \cup F) \leqslant 1$, we obtain

$$\begin{aligned} P(EF) &= P(E) + P(F) - P(E \cup F) \\ &\geqslant P(E) + P(F) - 1. \end{aligned} \tag{2}$$

Question 2

Let X be a random variable, prove that $E[X^2] \geqslant (E[X])^2$. When do we have equality?

Solution. Note that

$$E[(X - E(X))^2] \geqslant 0, \tag{3}$$

and

$$E[(X - E(X))^2] = E(X^2) - E(X)^2. \tag{4}$$

Therefore,

$$E(X^2) \geqslant E(X)^2. \tag{5}$$

When and only when $E(X) = X$ a.s., the equality holds.

Question 3

Let c be a constant and X be a random variable. Show that

- $\operatorname{Var}(cX) = c^2 \operatorname{Var}(X)$
- $\operatorname{Var}(c + X) = \operatorname{Var}(X)$

Solution. It is obvious that $E(cX) = cE(X), E(cX)^2 = cE(X)^2$, and $E(X+c) = E(X)$. As $\text{Var}(X) = E(X^2) - E(X)^2$, the results follow.

Show that if X is nonnegative continuous random variable with distribution F, then

$$\text{E}[X] = \int_0^\infty (1 - F(x))dx. \tag{6}$$

For a general continuous random variable X with finite mean, use $X = X^+ - X^-$ to prove

$$\text{E}[X] = \int_0^\infty (1 - F(x))dx - \int_{-\infty}^0 F(x)dx \tag{7}$$

Solution. Denote that $\bar{F}(x) = 1 - F(x)$, we have

$$\bar{F}(x) = \int_x^\infty f(t)dt. \tag{8}$$

Therefore,

$$\begin{aligned}\int_0^\infty \bar{F}(x)dx &= \int_0^\infty \int_x^\infty f(t)dtdx \\ &= \int_0^\infty \int_0^t f(t)dxdt, \ (X\text{is nonnegative}) \\ &= \int_0^\infty tf(t)dt \\ &= E(X).\end{aligned} \tag{9}$$

Besides, for a general continuous random variable X with the finite mean, use $X = X^+ - X^-$ and equation (9) we have

$$\begin{aligned}E[X] &= E[X^+] - E[X^-] \\ &= \int_0^\infty (1 - F_{X^+}(x))dx - \int_0^\infty (1 - F_{X^-}(x))dx.\end{aligned} \tag{10}$$

Also, $1 - F_{X^+}(x) = 1 - F(x), x \geqslant 0$, and,

$$1 - F_{X^-}(x) = P(X^- > x) = P(X < -x) = F(-x), x \geqslant 0, \tag{11}$$

we obtain

$$\begin{aligned}E(X) &= \int_0^\infty (1 - F(x))dx - \int_0^\infty (1 - F_{X^-}(x))dx \\ &= \int_0^\infty (1 - F(x))dx + \int_0^\infty (F(-x))d(-x) \\ &= \int_0^\infty (1 - F(x))dx - \int_{-\infty}^0 (F(x))dx.\end{aligned} \tag{12}$$

Till now, we have completed this solution.

Question 5

Prove that if random variables X and Y are jointly continuous, then $\mathrm{E}[X] = \int_{-\infty}^{\infty} \mathrm{E}[X|Y = y] f_Y(y) dy$

Solution. We have

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{E}[X|Y = y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x, y) dy dx \\ &= \int_{-\infty}^{+\infty} x f_X(x) dx = E(X) \end{aligned} \quad . \tag{13}$$

The solution has been completed.

Question 6

Let X_1 and X_2 be independent exponential random variables, each having rate μ. Let

$$X_{(1)} = \min\{X_1, X_2\} \text{ and } \mathrm{X}_{(2)} = \max\{X_1, X_2\}. \tag{14}$$

Find

- $\mathrm{E}[X_{(1)}]$ and $\mathrm{Var}(X_{(1)})$,
- $\mathrm{E}[X_{(2)}]$ and $\mathrm{Var}(X_{(2)})$

Solution. Note that

$$\begin{aligned} P(\min\{X_1, X_2\} > x) =& P(X_1 > x) P(X_2 > x) \\ =& e^{-\mu x} e^{-\mu x} \\ =& e^{-2\mu x}. \end{aligned} \quad , \tag{15}$$

we have

$$f_{X_{(1)}}(x) = 2\mu e^{-2\mu x} \sim \mathrm{Exp}(2\mu). \tag{16}$$

Therefore, $\mathrm{E}[X_{(1)}] = 1/2\mu$ and $\mathrm{Var}[X_{(1)}] = 1/(4\mu^2)$ follow.

Similarly,

$$\begin{aligned} P(\max\{X_1, X_2\} \leqslant x) =& P(X_1 \leqslant x) P(X_2 \leqslant x) \\ =& (1 - e^{-\mu x})(1 - e^{-\mu x}) \\ =& 1 - 2e^{-\mu x} + e^{-2\mu x} \end{aligned} \tag{17}$$

We have

$$f_{X_{(2)}}(x) = 2\mu e^{-\mu x} - 2\mu e^{-2\mu x}. \tag{18}$$

Then $\mathrm{E}[X_{(2)}] = \frac{3}{2\mu}$ and $\mathrm{Var}[X_{(2)}] = \frac{7}{4\mu^2}$ follow.

Question 7

Two individuals, A and B, both require kidney transplants. If she does not receive a new kidney, then A will die after an exponential time with rate μ_A, and B after an exponential time with rate μ_B. New kidneys arrive in accordance with a Poisson process having rate λ. It has been decided that the first kidney will go to A (or to B if B is alive and A is not at that time) and the next one to B (if still living).

- (a) What is the probability that A obtains a new kidney?
- (b) What is the probability that neither A nor B obtains a new kidney?
- (c) What is the probability that both A and B obtain new kidneys?

Solution. Let X_0 denote the arrived time of the first kidney, X_1 denote the time of death for A, and X_2 for B, and they are independent exponential random variables with rate λ, μ_A, μ_B.

Then the probability that the problem (a) is valid is equivalent to

$$\begin{aligned} P(X_0 < X_1) &= \int_0^\infty P\{X_0 < X_1 | X_0 = x\}\lambda e^{-\lambda x} dx \\ &= \frac{\lambda}{\lambda + \mu_A} \end{aligned} \tag{19}$$

Also, the results for (b) is followed as

$$\begin{aligned} P(X_0 < \max\{X_1, X_2\}) &= \int_0^\infty P(X_0 < \max\{X_1, X_2\} | X_0 = x\}\lambda e^{-\lambda x} dx \\ &= \int_0^\infty (e^{-\mu_A x} + e^{-\mu_B x} - e^{-(\mu_A + \mu_B)x})\lambda e^{-\lambda x} dx \\ &= \frac{\lambda}{\lambda + \mu_A} + \frac{\lambda}{\lambda + \mu_B} - \frac{\lambda}{\lambda + \mu_A + \mu_B}. \end{aligned} \tag{20}$$

The result for (c) is equivalent to

$$\begin{aligned} & P(N(X_1) = 1, N(X_2) = 2) \\ =& P(N(t) = 1, N(X_2) = 2 | X_1 = t) \\ =& \int_0^\infty P(N(t) = 1 | X_1 = t) P(N(X_2 - t) = 1 | X_1 = t) dt \\ =& \int_0^\infty \int_0^\infty P(N(t) = 1 | X_1 = t) P(N(s - t) = 1 | X_1 = t, X_2 = s) P(X_1 = t) P(X_2 = s) ds dt \\ =& \int_0^\infty \int_t^\infty P(N(t) = 1 | X_1 = t) P(N(s - t) = 1 | X_1 = t, X_2 = s) P(X_1 = t) P(X_2 = s) ds dt \\ =& \int_0^\infty (1 - e^{-\lambda t}) \mu_A e^{-\mu_A t} \int_t^\infty (1 - e^{-\lambda(s-t)}) \mu_B e^{-\mu_B s} ds dt \\ =& \mu_A \mu_B \int_0^\infty (1 - e^{-\lambda t}) e^{-\mu_A t} (\frac{1}{\mu_B} - \frac{1}{\mu_B + \lambda}) e^{-\mu_B t} dt \\ =& \frac{\lambda^2 \mu_A}{(\mu_A + \mu_B)(\mu_B + \lambda)(\mu_A + \mu_B + \lambda)} \end{aligned} \tag{21}$$

Let $\{N(t), t \geqslant 0\}$ be a Possion process with rate λ. For $i \leqslant n$ and $s < t$,

- (a) find $P\{N(t) = n | N(s) = i\}$;
- (b) find $P\{N(s) = i | N(t) = n\}$;

Solution. Considering (a), we find that

$$\begin{aligned} P\{N(t) = n | N(s) = i\} &= P\{N(t-s) = n-i\} \\ &= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-i}}{(n-i)!} \end{aligned} \tag{22}$$

Considering (b), we find that

$$\begin{aligned} P\{N(s) = i | N(t) = n\} &= \frac{P\{N(s) = i, N(t) = n\}}{P\{N(t) = n\}} \\ &= \frac{P\{N(s) = i\} P\{N(t-s) = n-i\}}{P\{N(t) = n\}} \\ &= \binom{n}{i} \frac{s^i (t-s)^{n-i}}{t^n}. \end{aligned} \tag{23}$$

The solution has been completed.

Question 9

Suppose that people immigrate into a territory according to a Poisson process with rate $\lambda = 2$ per day.

- (a) Find the probability these are exactly 12 arrivals in the following week(7 days).
- (b) Find the expected number of days until there have been 20 arrivals.
- (c) Given that there are 4 arrivals in a given two days, what is the probability that all 4 arrivals in the first one day.
- (d) Compute $\mathrm{E}[S_4 | N(2) = 2]$

Solution.

(a) $P\{N(7) = 12\} = e^{-14} \frac{14^{12}}{12!}$.

(b) $E[S_{20}] = 20 \times \frac{1}{2} = 10$.

(c)

$$\begin{aligned} P\{N(1)=4|N(2)=4\} &= \frac{P\{N(1)=4, N(2)=4\}}{P\{(N(2)=4\}} \\ &= \frac{P\{N(1)=4, N(2)-N(1)=0\}}{P\{N(2)=4\}} \\ &= \frac{P\{N(1)=4\}P\{N(1)=0\}}{P\{N(2)=4\}} \\ &= \frac{1}{16} \end{aligned}$$

(d)

$$\begin{aligned} \mathrm{E}[S_4|N(2)=2] &= 2+\mathrm{E}[S_2] \\ &= 3 \end{aligned}$$

Question 10

If $\{N(t), t \geqslant 0\}$ is a renewal process and $S_n = \sum_{i=1}^{n} X_i$. Is it true that

- (a) $N(t) < n$ if and only if $S_n > t$?
- (b) $N(t) \leqslant n$ if and only if $S_n \geqslant t$?
- (c) $N(t) > n$ if and only if $S_n < t$?

Solution. (a) is true.

(b) is not true. $S_n \geqslant t$ implies $N(t) \leqslant n$ while the reverse is not true.

(c) is not true. $N(t) > n$ implies $S_n < t$. While $S_n < t$ can not imply $N(t) > n$.