Academy of Mathematics and Systems Science Apr. 12, 2024
Chinese Academy of Sciences Assignment
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Stochastic Operations Research-Homeworks 2

Question 1

A Markov chain $\{X_n, n \geqslant 0\}$ with states 0,1,2, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

1. If $\mathrm{P}\{X_0 = 0\} = \mathrm{P}\{X_0 = 1\} = \frac{1}{4}$, find $\mathrm{E}[X_2]$.
2. Compute the mean of the sojourn time at state 0.
3. What are the limiting probabilities.
4. Compute the expected number of transitions need to return to state 0. (i.e., the mean recurrence time of state 0).

Solution.

1. We have

$$\boldsymbol{P}^{(2)} = \boldsymbol{P}^2 = \begin{bmatrix} \frac{1}{3} & \frac{5}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{9} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} [P\{X_2 = 0\}, P\{X_2 = 1\}, P\{X_2 = 2\}] &= [P\{X_0 = 0\}, P\{X_0 = 1\}, P\{X_0 = 2\}]\boldsymbol{P}^2 \\ &= [\frac{5}{12}, \frac{13}{72}, \frac{29}{72}]. \end{aligned}$$

Then we get

$$\mathrm{E}[X_2] = 1 \times \frac{13}{72} + 2 \times \frac{29}{72} = \frac{71}{72}.$$

2.

$$\mathrm{E}[R_0] = \frac{1}{1 - p_{ii}} = 2.$$

3. The limiting distribution equals to the stable distribution, which means that

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2}. \end{bmatrix}. \tag{1}$$

We also have

$$x_1 + x_2 + x_3 = 1. \tag{2}$$

Combine (1) and (2), we finally get

$$x_1 = \frac{2}{5}, x_2 = \frac{1}{5}, x_3 = \frac{2}{5}.$$

So the limiting probabilities are

$$[\frac{2}{5}, \frac{1}{5}, \frac{2}{5}]$$

4. Given that the Markov chain is positive recurrent, irreducible, aperiodic, we get that the stationary distribution is equivalent to the limiting distribution and

$$m_{00} = \frac{1}{\frac{2}{5}} = \frac{5}{2}$$

Question 2

1. Write down the forward equations for the pure birth process and prove that

$$P_{ii}(t) = e^{-\lambda_i t}, i \geqslant 0$$
$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds, j \geqslant i + 1.$$

2. Write down the forward equations for the birth and death process.

1. The forward equation can be formulated as

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t), j \geqslant i + 1;$$
$$p'_{i,j}(t) = -\lambda_j p_{i,j}(t), j = i;$$
$$p_{i,j}(t) = 0, 0 \leqslant j < i.$$

So we have

$$p'_{i,i}(t) = -\lambda_i p_{i,i}(t)$$

with $p_{i,i}(0) = 1$. Therefore,

$$p_{ii}(t) = e^{-\lambda_i t}, i \geqslant 0.$$

And by solving $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t), j \geqslant i + 1$, we have

$$(e^{\lambda_j t} p_{i,j}(t))' = \lambda_{j-1} e^{\lambda_j t} p_{i,j-1}(t). \tag{3}$$

Integrating both sides of (3) yields

$$e^{\lambda_j t} p_{i,j}(t) = \lambda_{j-1} \int_0^t e^{\lambda_j s} p_{i,j-1}(s) ds,$$

and we get the results

$$p_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} p_{i,j-1}(s) ds, j \geqslant i+1.$$

2.

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - (\lambda_j + \mu_j) p_{i,j}(t) + \mu_{j+1} p_{i,j+1}(t), j \geqslant 1;$$
$$p'_{i,0}(t) = -\lambda_0 p_{i,0}(t) + \mu_1 p_{i,1}(t).$$

Question 3

Consider a continuous-time Markov chain with infinitesimal generator matrix

$$\begin{bmatrix} -4 & 4 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ 0 & 2 & -4 & q_1 \\ 0 & 0 & 1 & q_2 \end{bmatrix}$$

1. What are q_1 and q_2?
2. Compute the limiting probabilities.

Solution.

1. We have
$$\begin{cases} 2 + (-4) + q_1 = 0 \\ 1 + q_2 = 0. \end{cases}$$
Therefore, $q_1 = 2, q_2 = -1$.

2. For a finite, irreducible, continuous-time Markov chain, the limiting distribution always exists and is identical to the stationary distribution of the chain. Therefore, We have $\boldsymbol{\pi} \boldsymbol{Q} = 0$, and $\sum_i \boldsymbol{\pi}_i = 1$. Thus,
$$\boldsymbol{\pi}_1 = 0.12, \boldsymbol{\pi}_2 = 0.16, \boldsymbol{\pi}_3 = 0.24, \boldsymbol{\pi}_4 = 0.48$$