1. 解 对于每一组 (k_i, c_i)，都可计算得

$$C^*(k_i, c_i) = \begin{cases} \dfrac{k_i\lambda}{Q_i} + \dfrac{hQ_i}{2} + c_i\lambda & Q_i > \sqrt{\dfrac{2k_i\lambda}{h}} \\ \sqrt{2\lambda h k_i} + c_i\lambda & Q_i < \sqrt{\dfrac{2k_i\lambda}{h}} < Q_{i+1} \\ \dfrac{k_i\lambda}{Q_{i+1}} + \dfrac{hQ_{i+1}}{2} + c_i\lambda & \sqrt{\dfrac{2k_i\lambda}{h}} \geq Q_{i+1} \end{cases}$$

取出 $\underset{i}{\operatorname{argmax}}\, C^*(k_i, c_i)$，然后再找出最优决策 Q_i 即可.

2. Suppose, in the newsvendor model (basic model), that demand D is discrete in the sense that there exists a sequence of real numbers, $\{x_1, x_2, \ldots\}$, representing all potential demand quantities, and $0 < x_1 < x_2 < \cdots$. Assume that $\mathbb{P}(D = x_i) = q_i$, $i = 1, 2, \ldots$. Furthermore, suppose all these quantities are admissible stock levels. Prove that the smallest x_i such that

$$\Phi(x_i) \geqslant \frac{p-c}{p}$$

is an optimal stock level.
Hint: In two steps:

(a) First show that for any $y \in (x_i, x_{i+1})$, $\mathcal{V}(y) \leqslant \max\{\mathcal{V}(x_i), \mathcal{V}(x_{i+1})\}$, $i = 0, 1, 2, \ldots$, where $x_0 = 0$;

(b) Second show that the smallest x_i such that $\mathcal{V}(x_{i+1}) - \mathcal{V}(x_i) \leq 0$ is an optimal stock level.

(a) 解 当 $y \in (x_i, x_{i+1})$ 时

$$\begin{aligned} V(y) &= (p-c)E(D-y) + pE(D-y)^- \\ &= \sum_{j=1}^{\infty}(p-c)q_j(x_j - y) + \sum_{j=1}^{i} pq_j(y - x_j) \\ &= \left(\sum_{j=1}^{\infty}(p-c)q_j - \sum_{j=1}^{i} pq_j\right)x_j + \left(\sum_{j=1}^{i} pq_j - \sum_{j=1}^{\infty}(p-c)q_j\right)y \\ &= \left(p - c - \sum_{j=1}^{i} pq_j\right)x_j + \left(\sum_{j=1}^{i} pq_j - (p-c)\right)y \end{aligned}$$

若 $\sum_{j=1}^{i} pq_j - (p-c) > 0$ 则 $V(y) \leq V(x_{i+1})$ 否则 $V(y) \leq V(x_i)$

(b) 解：由于 $\sum_{j=1}^{i} pq_j - (p-c)$ $\downarrow$，故(a)得证.

若 $\sum_{j=1}^{i} pq_j - (p-c)$ 首次 ≥ 0 则 $\sum_{j=1}^{i} pq_j - (p-c) = p\Phi(x_i) - (p-c) \geq 0$

$\Rightarrow \Phi(x_i) \geq \dfrac{p-c}{p}$

3. Suppose that g is a convex function defined on $\mathbb{R}$ and the real valued function G is defined on $\mathbb{R}$ by $G(y) := \mathbb{E}[g(y-D)]$, where D is a random variable with a density ϕ. Prove, without assuming that g is differentiable everywhere, that G is convex on $\mathbb{R}$.

解 $G(\lambda_1 y+\lambda_2 y) = E[g(\lambda_1 y+\lambda_2 y - D)] = E[g(\lambda_1 y - \lambda_1 D + \lambda_2 y - \lambda_2 D)]$

$\leq E[\lambda_1 g(y-D) + \lambda_2 g(y-D)]$

$= \lambda_1 E[g(y-D)] + \lambda_2 E[g(y-D)]$. ☒

4. The dynamic inventory problem of Section 4.1 assume that all shortages were backlogged. Assume now that the unsatisfied demand are lost (i.e., lost-sales model). A unit penalty cost of p is charged for each unit that is lost in each period. Note that the salvage value function $v_T(x) = -cx$ is now defined only for $x \geqslant 0$. All other aspects of the problem remain unchanged.

(a) Write down the optimality equations for this problem.

(b) Show that when $p+h > \alpha c$, a base stock policy with base stock level S is optimal in each period, where S is the following fractile

$$\Phi(S) = \frac{p-c}{p+h-\alpha c}. \tag{4.8}$$

(We assume all functions that will be used are differentiable)

(a) 解：一个周期内的费用为

$L(y) = h\int_0^y (y-\xi)\phi(\xi)d\xi + p\int_y^\infty (\xi-y)\phi(\xi)d\xi$

故最优方程为

$f_t(x) = \min_{y\geq x}\left\{c(y-x) + h\int_0^y (y-\xi)\phi(\xi)d\xi + p\int_y^\infty (\xi-y)\phi(\xi)d\xi + \alpha\int_0^y f_{t+1}(y-\xi)\phi(\xi)d\xi\right\}$

$t=1,2,\cdots,N$

$f_{N+1}(x) = \begin{cases} -cx & x\geq 0 \\ 0, & x<0 \end{cases}$

(b) 考虑第N个状态有

$f_t(x_N) = \min_{y_N\geq x_N}\left\{c(y_N-x_N) + h\int_0^{y_N}(y_N-\xi)\phi(\xi)d\xi + p\int_{y_N}^\infty(\xi-y_N)\phi(\xi)d\xi\right.$

$\left. - \alpha\int_0^{y_N} c(y_N-\xi)\phi(\xi)d\xi\right.$

记 $g(y)=c(y-x)+h\int_0^y (y-\xi)\phi(\xi)\,d\xi+p\int_y^\infty(\xi-y)\phi(\xi)\,d\xi-\alpha\int_0^y c(y-\xi)\phi(\xi)\,d\xi$

于是 $g'(y)=c+h\Phi(y)-p(1-\Phi(y))-\alpha c(\Phi(y))$

$g''(y)=(h+p-\alpha c)\Phi'(y)$ 　　由于 $h+p-\alpha c>0$ $\Phi'(y)\geq 0$

故g是一个凸函数, 当 $g'(y)=0$ 即 $\Phi(y)=\dfrac{p-c}{p+h-c}$ 时有最小值.

即第N个周期云订货至 $\Phi(S_N)=\dfrac{p-c}{p+h-c}$、下考虑前N-1个周期

注意到 $\left(h\int_0^y(y-\xi)\phi(\xi)\,d\xi+p\int_y^\infty(\xi-y)\phi(\xi)\,d\xi\right)''=(h+p)\Phi'(y)$

$h+p>\alpha c\Rightarrow h+p>0$ 故 $h\int_0^y(y-\xi)\phi(\xi)\,d\xi+p\int_y^\infty(\xi-y)\phi(\xi)\,d\xi$ 凸.

于是若 f_{t+1} 凸, $g_t(y):=c(y-x)+h\int_0^y(y-\xi)\phi(\xi)\,d\xi+p\int_y^\infty(\xi-y)\phi(\xi)\,d\xi+\alpha\int_0^y f_{t+1}(t-\xi)\phi(\xi)\,d\xi$ 凸. 则 $f_t(x)=\min_{y\geq x}\{g_t(y)\}$ 凸, 由 Lemma 4.1, g_t 的 minimizer(:=S) 是 base stock policy.

由 Theorem 4.1 知 A base stock policy 是最优的.

而 $g_t'(S)=c-p+h\Phi(S)+p\Phi(S)+\alpha\int_0^S f'_{t+1}(S-\xi)\phi(\xi)$

又 $f'_{t+1}(S-\xi)=(g_t(S)-c(S-\xi))'=-c$ 故

$g_t'(S)=c-p+h\Phi(S)+p\Phi(S)-\alpha c\Phi(S)=0\Rightarrow$ 满足 $\Phi(S)=\dfrac{p-c}{h+p-\alpha c}$ 的 S.

是t周期的最优策略 　　$\boxtimes$

5. proof Lemma 4.3

(1) f k凸 $\Rightarrow f((1-\lambda)x_1+\lambda x_2)\le(1-\lambda)f(x_1)+\lambda f(x_2)+\lambda k$

$\Rightarrow \alpha f((1-\lambda)x_1+\lambda x_2)\le \alpha(1-\lambda)f(x_1)+\alpha\lambda f(x_2)+\alpha\lambda k$

$= (1-\lambda)\alpha f(x_1)+\lambda\alpha f(x_2)+\alpha\lambda k$.

故 αf 是k凸的 $\forall k\ge\alpha k$

(2) f k凸, g k凸 $\Rightarrow$ $f((1-\lambda)x_1+\lambda x_2)\le(1-\lambda)f(x_1)+\lambda f(x_2)+\lambda k$ ①

$g((1-\lambda)x_1+\lambda x_2)\le(1-\lambda)g(x_1)+\lambda g(x_2)+\lambda k$ ②

①+②上得

(3) $G((1-\lambda)y_1+\lambda y_2)=E[v((1-\lambda)y_1+\lambda y_2-D)]\le E[(1-\lambda)v(y_1-D)+\lambda v(y_2-D)+\lambda k]$

$=(1-\lambda)G(y_1)+\lambda G(y_2)+\lambda k$

(4) 设 $z=(1-\lambda)x+\lambda y$

$\Rightarrow f(z)\le(1-\lambda)f(x)+\lambda f(y)+k=(1-\lambda)[k+f(y)]+\lambda f(y)+k\le f(y)+k$

6. (a) 由定义显然 $\alpha f(x\vee x')+\alpha f(x\wedge x')=\alpha(f(x\vee x')+f(x\wedge x'))$

$\le\alpha(f(x)+f(x'))=\alpha f(x)+\alpha f(x')\quad\forall\alpha\ge 0$

(b) 由定义显然 $f(x\vee x')+f(x\wedge x')\le f(x)+f(x')$

$g(x\vee x')+g(x\wedge x')\le g(x)+g(x')$

$(f+g)(x\vee x')+(f+g)(x\wedge x')\le(f+g)(x)+(f+g)(x')$

(c) 由定义显然 $h(x\vee x')+h(x\wedge x')=E[f((x\vee x')-D)+f((x\wedge x')-D)]$

$=E[f((x-D)\vee(x'-D))+f((x-D)\wedge(x'-D))]$

$\le E[f(x-D)+f(x'-D)]=E[f(x-D)]+E[f(x'-D)]$

$=h(x)+h(x')$