1. Derive $W(t)$ and $w(t)$ (the total waiting time CDF and its density) as given by the equations (4)-(5).

解: $W(t)=\sum_{n=1}^{\infty}(1-\rho)\rho^{n}\int_0^t \frac{\mu(\mu x)^{n-1}}{(n-1)!}e^{-\mu x}dx$

$=\sum_{n=1}^{\infty}(1-\rho)\int_0^t \frac{\mu(\rho\mu x)^{n-1}}{(n-1)!}e^{-\mu x}dx$

$=(1-\rho)\int_0^t\sum_{n=1}^{\infty}\frac{\mu(\rho\mu x)^{n-1}}{(n-1)!}e^{-\mu x}dx$

$=(1-\rho)\int_0^t \mu e^{\rho\mu x}e^{-\mu x}dx$

$=(1-\rho)\mu\int_0^t e^{(\rho-1)\mu x}dx=(1-\rho)\mu\frac{1}{(\rho-1)\mu}e^{(\rho-1)\mu x}\Big|_0^t$

$=1-e^{(\lambda-\mu)t}$

进而 $w(t)=(W(t))'=(\mu-\lambda)e^{(\lambda-\mu)t}$

2. Customers arrive at a service center according to a Poisson process with a rate of one every 15 minutes. The service time is exponentially distributed and the average time is 10 minutes. Answer the following questions:

(a) What is the probability that an arriving customer will have to wait?

(b) What is the probability that there are (strictly) more than four customers in the system?

(a) 解: $\rho=\frac{\lambda}{\mu}=\frac{2}{3}$

则 $P(L\geq 1)=\rho=\frac{2}{3}$

(b) 解 $P(L\geq 5)=\rho^5=(\frac{2}{3})^5=\frac{32}{243}$

3. What effect does simultaneously doubling λ and μ have on L, L_q, W, and W_q in the $M/M/1$ queue.

解: $L=\sum_{n=0}^{\infty}(1-\rho)\rho^n\cdot n=(1-\rho)\rho\sum_{n=1}^{\infty}(n\cdot\rho^{n-1})=(1-\rho)\rho\cdot(\sum_{n=1}^{\infty}\rho^n)'$

$=(1-\rho)\rho\left(\frac{\rho}{1-\rho}\right)'$

$=\frac{\rho}{1-\rho}=\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}=\frac{\lambda}{\mu-\lambda}$

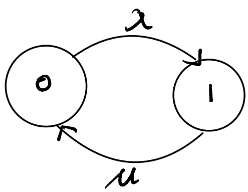
若 $\lambda'=\lambda\times 2$ $\mu'=\mu\times 2$ 则 $L'=L$ $L_q'=L'-\rho=L-\rho=\frac{\rho}{1-\rho}-\rho$

$=\frac{\rho^2}{1-\rho}$

$W'=\frac{L'}{\lambda'}=\frac{\frac{\lambda}{\mu-\lambda}}{2\lambda}=\frac{1}{2(\mu-\lambda)}$ $W=\frac{L_q'}{\lambda'}$

$=\frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})\cdot 2\lambda}=\frac{\frac{\lambda^2}{\mu^2}}{\frac{\mu-\lambda}{\mu}}\times\frac{1}{2\lambda}=\frac{\lambda}{2\mu(\mu-\lambda)}$

4. Obtain the performance measures L, L_q, W, and W_q in the $M/M/1/1$ queue.



$Q=\begin{pmatrix}-\lambda & \lambda \\ \mu & -\mu\end{pmatrix}$ 于是 $\pi=\left(\frac{\mu}{\mu+\lambda}, \frac{\lambda}{\mu+\lambda}\right)$.

这样 $L=\frac{\lambda}{\mu+\lambda}$ $\quad W=\frac{1}{\mu+\lambda}$

$L_q=0$ $\quad W_q=0$

5. Show the following:

(a) An $M/M/1$ is always better withe respect to L than an $M/M/2$ with the same ρ (note that $\rho=\lambda/\mu$ in $M/M/1$ and $\rho=\lambda/2\mu$ in $M/M/2$)

(b) An $M/M/2$ is always better than two independent $M/M/1$ queues with the same service rate but each getting half of the arrivals.

(1) 解: $L_1=\frac{\rho}{1-\rho}$ $\quad L_2=\frac{4\lambda\mu}{4\mu^2-\lambda^2}=\frac{4\lambda\mu\times\frac{1}{4\mu^2}}{1-\frac{\lambda^2}{4\mu^2}}=\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda^2}{4\mu^2}}=\frac{2\rho}{1-\rho^2}$

由于 $L_2=\frac{2\rho}{1-\rho^2}=\frac{\rho}{1-\rho}\times\frac{2}{1+\rho}$ 由于 $\frac{2}{1+\rho}>1$

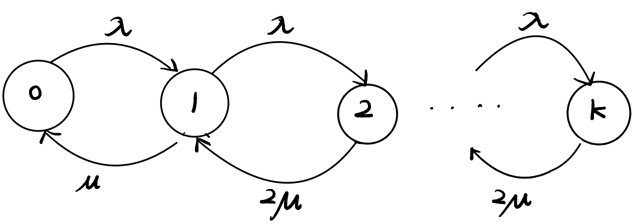
故L_2大于L_1。

(2) 两个独立的服务台 $L_1=\frac{2\times\frac{\lambda}{2}}{\mu-\frac{\lambda}{2}}=\frac{2\lambda}{2\mu-\lambda}$

一个队两个服务台 $L_2=\frac{4\lambda\mu}{4\mu^2-\lambda^2}$

此时 $\frac{4\lambda\mu}{4\mu^2-\lambda^2}=\frac{2\lambda}{2\mu-\lambda}\cdot\frac{2\mu}{2\mu+\lambda}<\frac{2\lambda}{2\mu-\lambda}$.

6. Obtain the steady-state distribution $\{\pi_n, 0\leq n\leq K\}$ of the number of customer in the system of $M/M/2/K$ queue with arrive rate λ and service rate μ for each server.



所以我们得到

$$\lambda\pi_0 = \mu\pi_1$$
$$(\lambda+\mu)\pi_1 = \lambda\pi_0 + 2\mu\pi_2$$
$$(\lambda+2\mu)\pi_2 = \lambda\pi_1 + 2\mu\pi_3$$
$$\vdots$$
$$(\lambda+2\mu)\pi_{k-1} = \lambda\pi_{k-2} + 2\mu\pi_k$$
$$2\mu\pi_k = \lambda\pi_{k-1}$$
$$\pi_0 + \pi_1 + \cdots + \pi_k = 1$$

$$\Rightarrow \begin{cases} \pi_1 = \dfrac{\lambda}{\mu}\pi_0 \\ \pi_2 = \dfrac{\lambda^2}{2\mu^2}\pi_0 \\ \pi_3 = \dfrac{\lambda^3}{4\mu^3}\pi_0 \\ \vdots \\ \pi_{k-1} = \dfrac{\lambda^{k-1}}{2^{k-2}\mu^{k-1}}\pi_0 \\ \pi_k = \dfrac{\lambda^k}{2^{k-1}\mu^k}\pi_0 \end{cases}$$

记 $\dfrac{\lambda}{2\mu} = \rho$

则 $\pi_0 = \dfrac{1-\rho}{1+\rho-2\rho^{k+1}}$

$$\pi_j = \frac{2\rho^j - 2\rho^{j+1}}{1+\rho-2\rho^{k+1}} \quad (j=1,2,\cdots,k).$$

7. Write the infinitesimal generator Q for the $M/E_r/1$ with system parameters $\lambda = 2$, $\mu = 3$, and $r = 2$.

8. Write the infinitesimal generator Q for the $E_r/M/1$ with system parameters $\lambda = 2$, $\mu = 3$, and $r = 2$.

7.

$$Q = \left(\begin{array}{c|cc|cc|cc|c}
-2 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline
0 & -8 & 6 & 2 & 0 & 0 & 0 & \cdots \\
6 & 0 & -8 & 0 & 2 & 0 & 0 & \cdots \\ \hline
0 & 0 & 0 & -8 & 6 & 2 & 0 & \cdots \\
0 & 6 & 0 & 0 & -8 & 0 & 2 & \cdots \\ \hline
0 & 0 & 0 & 0 & 0 & -8 & 6 & \cdots \\
0 & 0 & 0 & 6 & 0 & 0 & -8 & \cdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)$$

8.

$$
\left(\begin{array}{cc|cc|cc|c}
-4 & 4 & 0 & 0 & 0 & 0 & \cdots \\
0 & -4 & 4 & 0 & 4 & 0 & \cdots \\
\hline
3 & 0 & -7 & 4 & 0 & 0 & \cdots \\
0 & 3 & 0 & -7 & 4 & 0 & \cdots \\
\hline
0 & 0 & 3 & 0 & -7 & 4 & \cdots \\
0 & 0 & 0 & 3 & 0 & -7 & \cdots \\
\hline
\cdots & & \cdots & & \cdots & & \ddots
\end{array}\right)
$$

9. Let N_k be the number of customers in an $M/G/1$ queue just prior to the arrival of the nth customer. Explain why $\{N_k\}$ is not a discrete-time Markov chain.

解：

$$N_{k+1} = N_k - B_{k+1} + 1$$

其中 B_{k+1} 表示在第k个顾客到达至第k+1个顾客到达之间服务完成的数目。

但离开时间并非泊松过程，B_{k+1} 了解依赖于 $N_{k-1}, \cdots, N_1$

于是 N_{k+1} 并非离散马氏链.